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A GENERALIZATION OF THE MORITA-MUMFORD CLASSES TO EXTENDED MAPPING CLASS GROUPS FOR SURFACES

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ABSTRACT. Let $\Sigma_{g,1}$ be an orientable compact surface of genus g with 1 boundary component, and $\Gamma_{g,1}$ the mapping class group of $\Sigma_{g,1}$. We define a bigraded series of cohomology classes $m_{i,j} \in H^{2i+j-2}(\Gamma_{g,1}; \bigwedge^j H_1(\Sigma_{g,1};\mathbb{Z}))$, $2i+j-2 \geq 1$, $i,j \geq 0$. When $j=0$, the class $m_{i+1,0}$ is the i-th Morita-Mumford class [Mo][Mu]. It is proved that $H^r(\Gamma_{g,1}; \bigwedge^s H_1(\Sigma_{g,1};\mathbb{Q}))$ is generated by $m_{i,j}$'s for the case $r+s=2$ and the case $g \geq 5$ and $(r,s)=(1,3)$. Especially the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3] has an implicit representation by the classes $m_{0,3}$ and $m_{0,2}m_{1,1}$ over $\mathbb{Q}$.

INTRODUCTION

Let $g \geq 2$, $r,n \geq 0$ be integers. Let $\Sigma^n_{g,r}$ denote a 2-dimensional compact oriented C^∞ manifold (i.e., compact oriented surface) of genus g with r boundary components and (ordered) n punctures. The group of path-components $\pi_0(\mathrm{Diff}_+(\Sigma^n_{g,r}))$ is denoted by $\Gamma^n_{g,r}$ (or $\mathcal{M}^n_{g,r}$) and called the mapping class group of genus g with r boundary components and (ordered) n punctures. Here $\mathrm{Diff}_+(\Sigma^n_{g,r})$ denotes the topological group (endowed with C^∞ topology) consisting of all orientation preserving diffeomorphisms of $\Sigma^n_{g,r}$ which fix the boundary components and the punctures pointwise. When $n=0$, we drop the indices: $\Sigma_{g,r} = \Sigma^0_{g,r}$, $\Gamma_{g,r} = \Gamma^0_{g,r}$ and similarly $\Sigma_g = \Sigma^0_{g,0}$, $\Gamma_g = \Gamma^0_{g,0}$. Throughout this paper we denote by $H_1(\Sigma^n_{g,r})$ the first integral singular homology of the space $\Sigma^n_{g,r}$, on which the group $\Gamma^m_{g,s}$ act in an obvious way provided that $s \geq r$ and $m \geq n$.

By the *extended mapping class group* we mean the semi-direct product

$$\widetilde{\Gamma^n_{g,r}} := H_1(\Sigma_{g,1}) \rtimes \Gamma^n_{g,r}.$$

The purpose of the present paper is to define a bigraded series $\widetilde{m_{i,j}}$ of cohomology classes of the extended group $\widetilde{\Gamma_{g,1}}$, which is a generalization of the Morita-Mumford cohomology classes of the group Γ_g, and to investigate the ones of lower degree.

In §1 we prepare a theory of cohomology of pairs of groups, which is essential to the construction of the classes in the succeeding two sections. The E_2-term of the

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Lyndon-Hochschild-Serre spectral sequence of the group $\widetilde{\Gamma_{g,1}}$ with respect to the normal subgroup $H_1(\Sigma_{g,1})$ is given by

$$E_2^{p,q} = H^p(\Gamma_{g,1}; \bigwedge^q H^1(\Sigma_{g,1})).$$

So the classes $\widetilde{m_{i,j}}$ induce cohomology classes $m_{i,j}$ of the group $\Gamma_{g,1}$ with values in $\bigwedge^* H^1(\Sigma_{g,1})$. When $j = 0$, the class $m_{i+1,0}$ is the i-th Morita-Mumford class [Mo][Mu]. In §4, in order to see the non-triviality, we evaluate the classes $m_{0,2}$, $m_{1,1}$ and $m_{0,3}$ and prove that $H^r(\Gamma_{g,1}; \bigwedge^s H_1(\Sigma_{g,1}; \mathbb{Q}))$ is generated by $m_{i,j}$'s for the case $r+s=2$ (Proposition 4.1, Theorem 4.3, Corollary 4.5) and the case $g \geq 5$ and $(r,s) = (1,3)$ (Theorem 4.4). Especially the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3] has an implicit representation by the classes $m_{0,3}$ and $m_{0,2}m_{1,1}$ over $\mathbb{Q}$.

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Contents.

1. Cohomology of Pairs of Groups.

In this section we define cohomology groups $H^*(G, H : M)$ of a pair of groups (G, H) in the most naive sense. Denote by $C^*(G; M)$ the normalized cochain complex of a group G with values in a G-module M.

Let G be a group, H a subgroup of G, and M a G-module. We denote by $H^*(G, H; M)$ the cohomology group of the kernel of the restriction map

$$\mathrm{res} : C^*(G; M) \to C^*(H; M)$$

and call it *the cohomology group of the pair of groups (G, H) with values in the G-module M.* Since the restriction map res is surjective in the cochain level, we have a cohomology exact sequence

$$(1.1) \qquad \cdots \to H^{q-1}(H; M) \to H^q(G, H; M) \to H^q(G; M) \to H^q(H; M) \to \cdots,$$

In a natural way the cup product

$$\cup : H^*(G; M') \otimes H^*(G, H; M'') \to H^*(G, H; M' \otimes M'')$$

is defined.

Let $K \triangleleft G$ be a normal subgroup satisfying the condition

$$(1.2) \qquad HK = G.$$

Then we have the following Lyndon-Hochshild-Serre (LHS) spectral sequence [HS].

Proposition 1.3. *There is a spectral sequence converging to $H^*(G, H; M)$ whose E_2 term is given by*

$$E_2^{p,q} = H^p(G/K; H^q(K, K \cap H; M)).$$

It should be remarked how the quotient group G/K acts on the cohomology group $H^*(K, K \cap H; M)$. Since K is a normal subgroup of G, the group H acts on the normalized complex $C^*(K, K \cap H; M)$ by

$$(h \cdot c)(x_1, \ldots, x_n) := h(c(h^{-1}x_1 h, \ldots, h^{-1}x_n h)),$$

where $h \in H$, $c \in C^n(K, K\cap H; M)$ and $x_1, \ldots, x_n \in K$. For any element $h \in K \cap H$ consider a homotopy map

$$\Phi = \Phi_h : C^n(K, K \cap H; M) \to C^{n-1}(K, K \cap H; M)$$

given by

$$(\Phi_h c)(x_1, \ldots, x_{n-1}) := \sum_{j=0}^{n-1} (-1)^j c(x_1, \ldots, x_j, h, h^{-1}x_{j+1}h, \ldots, h^{-1}x_{n-1}h),$$

This map is well-defined and satisfies a homotopy equation

$$(d\Phi_h + \Phi_h d)c = h \cdot c - c \quad (\forall c \in C^*(K, K \cap H; M)).$$

Hence the subgroup $K \cap H$ acts on the cohomology group $H^*(K, K\cap H; M)$ <u>trivially</u>. From the condition (1.2) and the Second Isomorphism Theorem we have a natural isomorphism

$$G/K = H/K \cap H.$$

Thus the quotient group G/K acts on the cohomology group $H^*(K, K \cap H; M)$.

Let M, M_1 and M_2 be G/K-modules. Suppose

$$H^q(K, K \cap H; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } q = n, \\ 0, & \text{if } q > n. \end{cases} \tag{1.4}$$

Then the spectral sequence (1.3) induces a homomorphism

$$\pi_! : H^p(G, H; M) \to H^{p-n}(G/K; M), \tag{1.5}$$

which is called *the Gysin map* or *the fiber integral.* As usual we have

$$\pi_!(u \cup \pi^* v) = (\pi_! u) \cup v \in H^{p+q-n}(G/K; M_1 \otimes M_2), \tag{1.6}$$

for $u \in H^p(G, H; M_1)$ and $v \in H^q(G/K; M_2)$.

2. Mapping Class Groups.

From now on we consider mainly the mapping class groups $\Gamma_{g,1}$ and $\Gamma^1_{g,1}$. First we remark that the surface $\Sigma^1_{g,1}$ is obtained by glueing the surfaces $\Sigma_{g,1}$ and $\Sigma^1_{0,2}$ along the boundaries. So the diffeomorphism of $\Sigma_{g,1}$ is naturally extended to that of $\Sigma^1_{g,1}$. The infinite cyclic group $\mathbb{Z}$ acts on the surface $\Sigma^1_{0,2}$ by rotating the puncture and fixing the boundaries pointwise. Similarly this action is extended to that on $\Sigma^1_{g,1}$ in a natural way. Thus we obtain a natural homomorphism $\Gamma_{g,1} \times \mathbb{Z} \to \Gamma^1_{g,1}$, which is injective (see [I] §5). In the sequal we regard the group $\Gamma_{g,1} \times \mathbb{Z}$ as a subgroup of $\Gamma^1_{g,1}$ through the injection. Especially we may consider the cohomology group $H^*(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M)$ for an arbitrary $\Gamma^1_{g,1}$-module M. By forgetting the puncture we obtain an extension

$$1 \to \pi_1(\Sigma_{g,1}) \to \Gamma^1_{g,1} \xrightarrow{\pi} \Gamma_{g,1} \to 1. \tag{2.1}$$

Next we prepare a cycle induced by the "fiber" $\pi_1(\Sigma_{g,1})$. Choose a usual symplectic generator system of the fundamental group $\pi_1(\Sigma_{g,1})$:

$$a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g.$$

The loop on the boundary induces an element of $\pi_1(\Sigma_{g,1})$

$$w := \prod_{i=1}^{g} [a_i b_i], \quad [a_i, b_i] = a_i b_i {a_i}^{-1} {b_i}^{-1}.$$

We identify the group $\mathbb{Z}$ with the subgroup generated by w in $\pi_1(\Sigma_{g,1})$, and consider the cohomology group of the pair $H^*(\pi_1(\Sigma_{g,1}), \mathbb{Z})$.

Following Meyer [Me], we construct a normalized bar 2-chain $[\Sigma_{g,1}, \partial]$ as follows. For $1 \leq j \leq 4g$ let $w_j = {a_i}^{\pm 1}, {b_i}^{\pm 1}$ be the j-th generator in the element w, and $\widetilde{w_j} := w_1 w_2 \cdots w_j = a_1 b_1 \cdots w_j$. Let $\widetilde{w_0} = 1$. We define

$$[\Sigma_{g,1}, \partial] := \sum_{j=1}^{4g} [\widetilde{w_{j-1}} | w_j] - \sum_{i=1}^{g} ([a_i | {a_i}^{-1}] + [b_i | {b_i}^{-1}]) \in C_2(\pi_1(\Sigma_{g,1})). \tag{2.2}$$

Lemma 2.3. *For any trivial $\pi_1(\Sigma_{g,1})$-module M, we have*

$$H^*(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M) = \begin{cases} H \otimes M, & \text{if } * = 1, \\ M, & \text{if } * = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $H = H_1(\Sigma_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. The evaluation

$$< \cdot, [\Sigma_{g,1}, \partial] >: H^2(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M) \to M$$

is a well-defined isomorphism.

The first half of the lemma follows form the exact sequence (1.1), and the second from straightforward calculations.

Now let M be a $\Gamma_{g,1}$-module. The condition (1.2) is satisfied for our case $G = \Gamma^1_{g,1}$, $H = \Gamma_{g,1} \times \mathbb{Z}$ and $K = \pi_1(\Sigma_{g,1})$. It follws from Proposition 1.3 there exists a spectral sequence converging to

$$H^*(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M),$$

whose E_2 term is given by

$$H^p(\Gamma_{g,1}; H^q(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M)) = \begin{cases} H^p(\Gamma_{g,1}; H \otimes M), & \text{if } * = 1, \\ H^p(\Gamma_{g,1}; M), & \text{if } * = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence it induces a Gysin exact sequence

$$\cdots \to H^{q-1}(\Gamma_{g,1}; M) \to H^{q+1}(\Gamma_{g,1}; H \otimes M)$$
$$\to H^{q+2}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1}; M) \to \cdots.$$

Here the homomorphism $\pi_!$ is *the fiber integral* introduced in (1.5).

The Gysin sequence splits as follows. The identity map $1_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ generates the cohomology group $H^1(\mathbb{Z}) \cong \mathbb{Z}$. Regard $1_{\mathbb{Z}}$ as an element of $H^1(\Gamma_{g,1} \times \mathbb{Z})$ through the natural projection $\Gamma_{g,1} \times \mathbb{Z} \to \mathbb{Z}$ and denote by θ the image of $1_{\mathbb{Z}}$ under the connecting homomorphism δ^*:

$$\theta := \delta^*(1_{\mathbb{Z}}) \in H^2(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; \mathbb{Z}).$$

Since $< \theta, [\Sigma_{g,1}, \partial] >= -1$, we have

$$\pi_! \theta = -1 \in H^0(\Gamma_{g,1}; \mathbb{Z}). \tag{2.4}$$

Thus, from the property (1.6) of the fiber integral $\pi_!$, the sequence splits. Consequently we have

Proposition 2.5. *For any $\Gamma_{g,1}$-module M, we have an exact sequence*

$$0 \to H^{q+1}(\Gamma_{g,1}; H \otimes M) \to H^{q+2}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1}; M) \to 0,$$

which splits as follows:

$$H^{q+2}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M) = H^{q+1}(\Gamma_{g,1}; H \otimes M) \oplus \theta \cup H^q(\Gamma_{g,1}; M).$$

On the other hand, taking the semi-direct product of the extension (2.1) and the $\Gamma_{g,1}$-module $H_1(\Sigma_{g,1}; \mathbb{Z})$, we have an extension of groups

$$1 \to \pi_1(\Sigma_{g,1}) \to \widetilde{\Gamma^1_{g,1}} \xrightarrow{\widetilde{\pi}} \widetilde{\Gamma^1_{g,1}} \to 1. \tag{2.6}$$

In a similar way to the fiber integral $\pi_!$ we obtain *the fiber integral*

$$\widetilde{\pi}_! : H^q(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z}) \to H^{q-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}).$$

3. Construction of Cohomology Classes.

For the rest we often abbreviate

$$H := H_1(\Sigma_{g,1};\mathbb{Z}) = H^1(\Sigma_{g,1};\mathbb{Z}).$$

The isomorphism on the right-hand side is the Poincaré duality, which is $\Gamma_{g,1}$-equivariant. We remark this H plays a different role in the sequal from the subgroup H in the preceeding sections.

Denote by $\cdot$ the intersection form on $H \cong H_1(\Sigma_g;\mathbb{Z})$.

Choose a simple curve l on $\Sigma^1_{g,1}$ connecting the puncture to a point on the boundary. Define a 2-cochain $\widetilde{\omega_l} \in C^2(\widetilde{\Gamma^1_{g,1}};\mathbb{Z})$ by

$$\widetilde{\omega_l}(u_1\gamma_1, u_2\gamma_2) := \gamma_1(\gamma_2 l - l)\cdot u_1, \quad u_1, u_2 \in H,\ \gamma_1,\gamma_2 \in \Gamma^1_{g,1}, \tag{3.1}$$

and a 1-cochain $\omega_l \in C^1(\Gamma^1_{g,1};H)$ by

$$\omega_l(\gamma) = \gamma l - l \in H, \quad \gamma \in \Gamma^1_{g,1}, \tag{3.2}$$

where we remark $\gamma_2 l - l$ can be regarded as a closed curve on $\Sigma_{g,1}$. A straightforward computation shows the cochains $\widetilde{\omega_l}$ and ω_l are cocycles. On the other hand, if $\gamma \in \Gamma_{g,1} \times \mathbb{Z}$, the curve $\gamma l - l$ is homotopic to a curve in the boundary $\partial\Sigma_{g,1}$. Hence $\gamma l - l = 0 \in H$. Thus we have

$$\widetilde{\omega_l} \in Z^2(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z};\mathbb{Z}) \quad \text{and} \quad \omega_l \in Z^2(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H). \tag{3.3}$$

To study the dependence of the cohomology classes $[\widetilde{\omega_l}]$ and $[\omega_l]$ on the choice of the curve l, choose another simple curve l' on $\Sigma^1_{g,1}$ connecting the puncture to the boundary. The cycle $v := l' - l$ on $\Sigma^1_{g,1}$ may be regarded as an element in H. So we have

$$\omega_{l'} - \omega_l = dv \in C^1(\Gamma^1_{g,1};H). \tag{3.4}$$

When we define a 1-cochain $c_v \in C^1(\widetilde{\Gamma^1_{g,1}})$ by

$$c_v(u\gamma) := (\gamma v)\cdot u, \quad u \in H, \gamma \in \Gamma^1_{g,1},$$

we have

$$\widetilde{\omega_{l'}} - \widetilde{\omega_l} = dc_v. \tag{3.5}$$

Let $e \in H^2(\Gamma^1_g;\mathbb{Z})$ be the Euler class of the central extension

$$1 \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma^1_g \to 1.$$

The class e may be regarded as a cohomology class in $H^2(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z};\mathbb{Z})$ in an obvious way. From (3.4) and (3.5), if $i + j \geq 2$, the products

$$e^i[\widetilde{\omega_l}]^j \in H^{2i+2j}(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z};\mathbb{Z}) \quad \text{and}$$
$$e^i[\omega_l]^j \in H^{2i+j}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; \textstyle\bigwedge^j H)$$

are independent of the choice of the curve l. We denote them by $e^i\widetilde{\omega}^j$ and $e^i\omega^j$ respectively.

Recall $H^p(\Gamma^1_{g,1}; \bigwedge^q H)$ is the $E_2^{p,q}$-term of the LHS spectral sequence of $\widetilde{\Gamma_{g,1}}$ with respect to the normal subgroup H. Since we have

$$\widetilde{\omega_l}(u_1, u_2\gamma_2) = \omega_l(\gamma_2) \cdot u_1$$

for $\forall u_1, u_2 \in H$ and $\gamma_2 \in \Gamma^1_{g,1}$, the class $[\omega_l] \in H^1(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H)$ is equal to that induced by $\widetilde{\omega_l} \in H^2(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z})$. Now we can define the cohomology classes $\widetilde{m_{i,j}}$ and $m_{i,j}$. Consider two extensions of groups

$$1 \to \pi_1(\Sigma_{g,1}) \to \Gamma^1_{g,1} \xrightarrow{\pi} \Gamma^1_{g,1} \to 1 \tag{2.1}$$

$$1 \to \pi_1(\Sigma_{g,1}) \to \widetilde{\Gamma^1_{g,1}} \xrightarrow{\widetilde{\pi}} \widetilde{\Gamma^1_{g,1}} \to 1. \tag{2.6}$$

We define

$$\begin{aligned} m_{i,j} &:= \pi_!(e^i\omega^j) \in H^{2i+j-2}(\Gamma_{g,1}; {\textstyle\bigwedge^j} H) \\ \widetilde{m_{i,j}} &:= \widetilde{\pi}_!(e^i\widetilde{\omega}^j) \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}) \end{aligned} \tag{3.6}$$

for $i, j \in \mathbb{N}$. Here $\pi_!$ and $\widetilde{\pi}_!$ are the fiber integrals introduced in the previous section. Clearly $m_{i+1,0}$ and $\widetilde{m_{i+1,0}}$ are equal to (the image of) the i-th Morita-Mumford (tautological) class $e_i (= \kappa_i) \in H^{2i}(\Gamma_g; \mathbb{Z})$ [Mo][Mu]:

$$m_{i+1,0} = \widetilde{m_{i+1,0}} = e_i \in H^{2i}(\Gamma_{g,1}; \mathbb{Z}). \tag{3.7}$$

Remark 3.8. Let $\mathcal{F}_{g-1}$ be the dressed moduli of pairs of compact Riemann surfaces of genus g and holomorphic line bundles of degree $g-1$ on the surfaces. The space $\mathcal{F}_{g-1}$ is aspherical and its π_1 is equal to $\widetilde{\Gamma_{g,1}}$. As is known, the Lie algebra of holomorphic differential operators "near S^1" has an infinitesimal and transitive action on the dressed moduli $\mathcal{F}_{g-1}$ [ADKP]. The $\widetilde{m_{i,j}}$'s have their origins in the equivariant cohomology of $\mathcal{F}_{g-1}$ under this action [Ka1].

4. Evaluations.

The purpose of this section is to evaluate the classes $m_{2,0}$, $m_{1,1}$ and $m_{0,3}$ and to prove that $H^r(\Gamma_{g,1}; \bigwedge^s H_1(\Sigma_{g,1}; \mathbb{Q}))$ is generated by $m_{i,j}$'s for the case $r+s=2$ and the case $g \geq 5$ and $(r,s)=(1,3)$.

Denote by Ω the symplectic form on H induced by the cup product:

$$\Omega := \sum_{i=1}^{g} a_i \otimes b_i - b_i \otimes a_i \in \bigwedge^2 H,$$

where $\{a_i, b_i; 1 \leq i \leq g\}$ is (the homology classes induced by) a symplectic generator system of the fundamental group $\pi_1(\Sigma_{g,1})$ as in §2.

Proposition 4.1.

$$m_{0,2} = \pi_!(\omega^2) = 2\Omega \in H^0(\Gamma_{g,1}; \bigwedge^2 H).$$

Proof. It suffices to show that

$$< \omega^2, [\Sigma_{g,1}, \partial] >= 2\Omega.$$

Here $[\Sigma_{g,1}, \partial]$ is a 2-chain introduced in (2.2). Since $\omega(\widetilde{w_{4i}}) = 0$, we have

$$< \omega^2, [\Sigma_{g,1}, \partial] >= \sum_{j=1}^{4g} \omega^2(\widetilde{w_{j-1}}, w_j) - \sum_{i=1}^{g} (\omega^2(a_i, {a_i}^{-1}) + \omega^2(b_i, {b_i}^{-1}))$$

$$= \sum_{i=1}^{g} a_i \wedge b_i - (a_i + b_i) \wedge a_i - (a_i + b_i - a_i) \wedge b_i + a_i \wedge a_i + b_i \wedge b_i$$

$$= \sum_{i=1}^{g} a_i \wedge b_i - b_i \wedge a_i = 2\Omega,$$

as was to be shown. □

Next we study the classes $m_{1,1}$ and $m_{0,3}$. In [Mo1] and [Mo2] Morita proved

$$H^1(\Gamma_{g,1}; H) = \mathbb{Z}, \quad \text{and} \quad H^1(\Gamma_{g,1}; \bigwedge^3 H) = \mathbb{Z}^2, \tag{4.2}$$

where we denote $H = H_1(\Sigma_{g,1}; \mathbb{Z})$ as before. Our results are

Theorem 4.3. *The class $m_{1,1}$ generates the group $H^1(\Gamma_{g,1}; H)$.*

Theorem 4.4. *If $g \geq 5$, the classes $m_{0,2}m_{1,1}$ and $m_{0,3}$ generate the group $H^1(\Gamma_{g,1}; \bigwedge^3 H \otimes \mathbb{Q})$.*

The rest of this section is devoted to the proof of the theorems. As was shown by Harer [H], if $g \geq 3$, we have $H^2(\Gamma_{g,1}; \mathbb{Q}) = \mathbb{Q}$ and the class $m_{2,0} = e_1$ generates it. Hence in the case $r+s=2$ the groups $H^r(\Gamma_{g,1}; \bigwedge^s H \otimes \mathbb{Q})$ are generated by the classes $m_{i,j}$'s. Consequently

Corollary 4.5. *If $g \geq 3$, the group $H^2(\widetilde{\Gamma_{g,1}}; \mathbb{Q})$ is isomorphic to $\mathbb{Q}^3$ and the classes $\widetilde{m_{0,2}}$, $\widetilde{m_{1,1}}$ and $\widetilde{m_{2,0}}$ form its free basis.*

The first half of the corollary has been already shown by Arbarello et. al.([ADKP] §5).

To prove the theorems we endow the surface Σ_g with a Riemannian metric. Fix a sufficiently small positive real ϵ. Let $\varpi : ST\Sigma_g \to \Sigma_g$ be the unit tangent bundle of the surface Σ_g. Denote by D^2 the unit disk in $\mathbb{C}$: $D^2 := \{z \in \mathbb{C}; |z| \leq 1\}$. We define a disk bundle D_g over $ST\Sigma_g$ by

$$D_g := \{(v_1, x_2) \in ST\Sigma_g \times \Sigma_g; \mathrm{dist}(\varpi(v_1), x_2) \leq \epsilon\},$$

The first projection induces its projection $p_1 : D_g \to ST\Sigma_g$. The disk bundle is trivial through the projection

$$ST\Sigma_g \times D^2 \to D_g, \quad (v, z) \mapsto (v, \mathrm{Exp}_{\varpi(v)}(\epsilon z v)).$$

Here we use the (almost) complex structure induced by the given Riemannian metric.

Consider a $\Sigma_{g,1}$-bundle

$$p_1 : Y_g (:= ST\Sigma_g \times \Sigma_g - \mathrm{int}\, D_g) \to ST\Sigma_g$$

induced by the first projection. The fundamental group $\pi_1(ST\Sigma_g)$ is embedded into the group $\Gamma_{g,1}$ through the classifying map ι of the bundle $p_1 : Y_g \to ST\Sigma_g$, and is identified with the kernel of the forgetting map $\Gamma_{g,1} \to \Gamma_g$:

$$1 \to \pi_1(ST\Sigma_g) \xrightarrow{\iota} \Gamma_{g,1} \to \Gamma_g \to 1.$$

Since the spaces Σ_g, $ST\Sigma_g$, D_g and Y_g are all aspherical, we drop the notations $\pi_1(\cdot)$ in the cohomology groups.

The identity map $1_H \in \mathrm{Hom}(H, H)$ induces a cohomology class

$$1_H \in H^1(\Sigma_g; H) \cong \mathrm{Hom}(H, H).$$

By abuse of notation we denote also by 1_H the pull-back $\varpi^*(1_H)$ through the projection $\varpi : ST\Sigma_g \to \Sigma_g$:

$$1_H = \varpi^*(1_H) \in H^1(ST\Sigma_g; H) \cong \mathrm{Hom}(H, H).$$

In [Mo1] Morita proved the following theorem (see also [Mo2] p.81 l.4 ff).

Theorem 4.6 (Morita).

$$H^1(\Gamma_{g,1}; H) = \mathbb{Z}.$$

Furthermore a crossed homomorphism $k : \Gamma_{g,1} \to H$ represents a generator of the group $H^1(ST\Sigma_g; H)$ if and only if the restriction of k to $\pi_1(ST\Sigma_g)$ is equal to $\pm(2-2g)1_H$:

$$\iota^*(k) = \pm(2-2g)1_H \in H^1(ST\Sigma_g; H).$$

As for $\bigwedge^3 H = \bigwedge^3 H_1(\Sigma_{g,1}; H)$ he proved the following ([Mo3] Theorem 5.1, see also the proof of Corollary 5.7). Let k_0 be a generator of the group $H^1(\Gamma_{g,1}; H)$.

Theorem 4.7 (Morita). *If $g \geq 3$,*

$$H^1(\Gamma_{g,1}; \bigwedge^3 H) = \mathbb{Z} \oplus \mathbb{Z}.$$

The class $\Omega \wedge k_0$ and a class he named $2\tilde{k}$ form its free basis. Furtheremore their restriction to $\pi_1(ST\Sigma_g)$ are given by

$$\iota^*(\Omega \wedge k_0) = \pm(2-2g)\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H),$$
$$\iota^*(2\tilde{k}) = 2\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H).$$

Therefore our theorems are reduced to

Assertion 4.8.

$$\iota^*(m_{1,1}) = -(2-2g)1_H \in H^1(ST\Sigma_g; H) \tag{1}$$

$$\iota^*(m_{0,3}) = -6\,\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H) \tag{2}$$

In fact, (1) implies Theorem 4.3 by Theorem 4.6. So we have $m_{2,0}m_{1,1} = \pm 2\Omega \wedge k_0$. From Theorem 4.7 the class $m_{0,3}$ has a representation $m_{0,3} = a\Omega \wedge k_0 + b(2\tilde{k})$ for some integers a and b. Since $H^1(ST\Sigma_g; \bigwedge^3 H) = H \otimes \bigwedge^3 H$ is $\mathbb{Z}$-free, we have

$$-6 = \pm a(2-2g) + 2b,$$

and so $b \equiv -3 \mod (g-1)$, while $g - 1 \geq 4$. Thus we have $b \neq 0$.

This completes the proof of Theorems 4.3 and 4.4 modulo Assertion 4.8.

Let M be a $\pi_1(ST\Sigma_g)$-module. By excision we may consider the map

$$j^* : H^*(Y_g, \partial Y_g; M) \underset{\text{exc.}}{\cong} H^*(ST\Sigma_g \times \Sigma_g, D_g; M) \to H^*(ST\Sigma_g \times \Sigma_g; M).$$

The fiber integral $p_{1!} : H^*(Y_g, \partial Y_g; M) \to H^{*-2}(ST\Sigma_g; M)$ decomposes itself into

$$H^*(Y_g, \partial Y_g; M) \xrightarrow{j^*} H^*(ST\Sigma_g \times \Sigma_g, D_g; M) \xrightarrow{p_{1!}} H^{*-2}(ST\Sigma_g; M).$$

Here the latter fiber integral $p_{1!}$ is the usual one induced by the first projecion $p_1 : ST\Sigma_g \times \Sigma_g \to ST\Sigma_g$. Thus we have

$$\iota^* m_{1,1} = p_{1!}j^*(e\omega) \quad \text{and} \quad \iota^* m_{0,3} = p_{1!}j^*(\omega^3).$$

Now we have

$$j^*(e) = p_2{}^*e' \in H^2(ST\Sigma_g \times \Sigma_g; \mathbb{Z})$$
$$j^*(\omega) = p_2{}^*1_H - p_1{}^*1_H \in H^1(ST\Sigma_g \times \Sigma_g; H),$$

where $p_2 : ST\Sigma_g \times \Sigma_g \to \Sigma_g$ is the second projection and

$$e' = e(T\Sigma_g) \in H^2(\Sigma_g; \mathbb{Z}).$$

Since $e'1_H \in H^3(\Sigma_g; H) = 0$, we have

$$\iota^* m_{1,1} = p_{1!}j^*(e\omega) = p_{1!}(p_2{}^*e')(p_2{}^*1_H - p_1{}^*1_H)$$
$$= -(p_{1!}p_2{}^*e')1_H = -(2-2g)1_H.$$

On the other hand, since $(1_H)^3 \in H^3(\Sigma_g; \bigwedge^3 H) = 0$ and $p_{1!}p_2{}^*1_H \in H^{-1}(ST\Sigma_g; H) = 0$, we have

$$j^*(\omega^3) = (p_2{}^*1_H - p_1{}^*1_H)^3 = -3(p_2{}^*(1_H)^2)p_1{}^*1_H + 3(p_2{}^*1_H)p_1{}^*(1_H)^2$$

and

$$p_{1!}j^*(\omega^3) = -3(p_{1!}p_2{}^*(1_H)^2)1_H + 3(p_{1!}p_2{}^*1_H)(1_H)^2 = -3 < (1_H)^2, [\Sigma_g] > 1_H,$$

where we denote by $[\Sigma_g] \in H_2(\Sigma_g; \mathbb{Z})$ the fundamental class. From a similar calculation to Proposition 4.1 follows $< (1_H)^2, [\Sigma_g] >= 2\Omega$. Therefore

$$\iota^* m_{0,3} = p_{1!}j^*(\omega^*) = -6\Omega \wedge 1_H.$$

This completes the proof of Assertion 4.8 and so those of Theorems 4.3 and 4.4.

Remark 4.9. The crossed homomorphism $\tilde{k} = \frac{1}{2}2\tilde{k} : \Gamma_{g,1} \to \frac{1}{2}\bigwedge^3 H$ in (4.7) is the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3]. Hence Theorem 4.4 implies the Johnson homomorphism $\tilde{k}$ is represented by $m_{0,3}$ and $m_{0,2}m_{1,1}$ over $\mathbb{Q}$. The author, however, doesn't know the explicit representation of $\tilde{k}$ by $m_{0,3}$ and $m_{0,2}m_{1,1}$.

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